Exercise 29

The equations for the current $I(x,t)$ and potential $V(x,t)$ at a point x and time t of a transmission line containing resistance R, inductance L, capacitance C, and leakage inductance G are

$$LI_t + RI = -V_x, \quad \text{and} \quad CV_t + GV = -I_x.$$

Show that both I and V satisfy the telegraph equation

$$\frac{1}{c^2}u_{tt} - u_{xx} + au_t + bu = 0,$$

where $c^2 = (LC)^{-1},\ a = LG + RC$, and $b = RG$.
Solve the telegraph equation for the following cases with $R = 0$ and $G = 0$:

(a) $V(x,t) = V_0 H(t)$ at $x = 0,\ t > 0,\ V(x,t) \to 0$ as $x \to \infty,\ t > 0$, where V_0 is constant.

(b) $V(x,t) = V_0 \cos \omega t$ at $x = 0,\ t > 0,\ V(x,t) \to 0$ as $x \to \infty,\ t > 0$.

Solution

We have

$$LI_t + RI = -V_x \tag{1}$$
$$CV_t + GV = -I_x. \tag{2}$$

Proof V Satisfies the Telegraph Equation

Differentiate both sides of equation (1) with respect to x and both sides of equation (2) with respect to t.

$$LI_{tx} + RI_x = -V_{xx}$$
$$CV_{tt} + GV_t = -I_{xt}.$$

If we assume that I_{tx} and I_{xt} are continuous, then they are equal by Clairaut's theorem. Multiply both sides of the second equation by -1 and replace I_{xt} with I_{tx}

$$LI_{tx} + RI_x = -V_{xx}$$
$$-CV_{tt} - GV_t = I_{tx}.$$

Substitute the expression for I_{tx} and substitute equation (2) for I_x into the first equation.

$$L(-CV_{tt} - GV_t) + R(-CV_t - GV) = -V_{xx}$$

Expand the left side and multiply both sides by -1.

$$LCV_{tt} + LGV_t + RCV_t + RGV = V_{xx}$$

Bring V_{xx} to the left side and factor V_t.

$$LCV_{tt} - V_{xx} + (LG + RC)V_t + RGV = 0$$

Let
$$\frac{1}{c^2} = LC \quad \text{and} \quad a = LG + RC \quad \text{and} \quad b = RG$$
so that we get
$$\frac{1}{c^2}V_{tt} - V_{xx} + aV_t + bV = 0.$$
Therefore, V satisfies the telegraph equation.

Proof I Satisfies the Telegraph Equation

Differentiate both sides of equation (1) with respect to t and both sides of equation (2) with respect to x.

$$LI_{tt} + RI_t = -V_{xt}$$
$$CV_{tx} + GV_x = -I_{xx}.$$

If we assume that V_{tx} and V_{xt} are continuous, then they are equal by Clairaut's theorem. Multiply both sides of the first equation by -1 and replace V_{xt} with V_{tx}

$$-LI_{tt} - RI_t = V_{tx}$$
$$CV_{tx} + GV_x = -I_{xx}.$$

Substitute the expression for V_{tx} and substitute equation (1) for V_x into the second equation.

$$C(-LI_{tt} - RI_t) + G(-LI_t - RI) = -I_{xx}$$

Expand the left side and multiply both sides by -1.

$$LCI_{tt} + RCI_t + LGI_t + RGI = I_{xx}$$

Bring I_{xx} to the left side and factor I_t.

$$LCI_{tt} - I_{xx} + (LG + RC)I_t + RGI = 0$$

Let
$$\frac{1}{c^2} = LC \quad \text{and} \quad a = LG + RC \quad \text{and} \quad b = RG$$
so that we get
$$\frac{1}{c^2}I_{tt} - I_{xx} + aI_t + bI = 0.$$
Therefore, I satisfies the telegraph equation.

Part (a)

When $R = 0$ and $G = 0$, $a = 0$ and $b = 0$, so the telegraph equation reduces to the wave equation.

$$\frac{1}{c^2}V_{tt} - V_{xx} = 0$$

The general solution to the wave equation is the sum of a wave travelling to the left and a wave travelling to the right.
$$V(x,t) = F(x + ct) + G(x - ct)$$

Plug in $x = 0$ and use the given boundary condition.

$$V(0,t) = F(ct) + G(-ct) = V_0 H(t)$$

Make the substitution,

$$s = ct \quad \to \quad t = \frac{s}{c},$$

to better see what the two unknown functions F and G should be.

$$F(s) + G(-s) = V_0 H\left(\frac{s}{c}\right)$$

The function on the right side is the form F and G will have. A factor of 1/2 needs to be in front of each so that when we add them, we get a coefficient of V_0. Replace s with $x + ct$ to get the form for F, and replace s with $-(x - ct)$ to get the form for G. The minus sign is included in front because we want $+t$ to be left over when we set $x = 0$.

$$V(x,t) = \frac{1}{2} V_0 H\left(\frac{x + ct}{c}\right) + \frac{1}{2} V_0 H\left(-\frac{x - ct}{c}\right)$$

Therefore, after factoring and splitting up the fractions,

$$V(x,t) = \frac{V_0}{2}\left[H\left(t + \frac{x}{c}\right) + H\left(t - \frac{x}{c}\right)\right].$$

Part (b)

When $R = 0$ and $G = 0$, $a = 0$ and $b = 0$, so the telegraph equation reduces to the wave equation.

$$\frac{1}{c^2} V_{tt} - V_{xx} = 0$$

The general solution to the wave equation is the sum of a wave travelling to the left and a wave travelling to the right.

$$V(x,t) = F(x + ct) + G(x - ct)$$

Plug in $x = 0$ and use the given boundary condition.

$$V(0,t) = F(ct) + G(-ct) = V_0 \cos \omega t$$

Make the substitution,

$$s = ct \quad \to \quad t = \frac{s}{c},$$

to better see what the two unknown functions F and G should be.

$$F(s) + G(-s) = V_0 \cos\left(\frac{\omega}{c} s\right)$$

The function on the right side is the form F and G will have. A factor of 1/2 needs to be in front of each so that when we add them, we get a coefficient of V_0. Replace s with $x + ct$ to get the form for F, and replace s with $-(x - ct)$ to get the form for G. The minus sign is included in front because we want $+t$ to be left over when we set $x = 0$.

$$V(x,t) = \frac{1}{2} V_0 \cos\left[\frac{\omega}{c}(x + ct)\right] + \frac{1}{2} V_0 \cos\left[-\frac{\omega}{c}(x - ct)\right]$$

Because cosine is an even function, the minus sign in front can be removed. Factor $V_0/2$ and distribute ω/c.

$$\frac{V_0}{2}\left[\cos\left(\frac{\omega x}{c}+\omega t\right)+\cos\left(\frac{\omega x}{c}-\omega t\right)\right]$$

Therefore, with the sum-to-product trigonometric identity for cosine,

$$V(x,t)=V_0\cos\omega t\cos\frac{\omega x}{c}.$$